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Note

A note on binary matroid with no $M(K_{3,3})$-minor

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Abstract

We prove that if M is an internally 4-connected binary matroid with an $M(K_5)$-minor and with no $M(K_{3,3})$-minor, then either M has rank 4, or M is isomorphic to one of the following matroids: T_{12}, T_{12}/e, $\widetilde{T_{11}}$, $\widetilde{T_{12}}$, and $\widetilde{T_{13}}$.

Keywords: Binary matroids; Splitter Theorem; Blocking sequence; Internal 4-connectivity

1. Introduction

We assume the reader is familiar with matroid theory. Our notation and terminology will follow [5]. We begin by recalling Hall's Theorem [2].

Theorem 1.1 *(Hall, 1943). If G is a simple 3-connected graph, then G has no $K_{3,3}$-minor if and only if G is planar or $G \cong K_5$.*

Theorem 1.1 strengthens Kuratowski's Theorem in the case of 3-connected graphs. It implies the only simple 3-connected graph with a K_5-minor and with no $K_{3,3}$-minor is the graph K_5 itself. We prove an analogue of Hall's Theorem for binary matroids in the internally 4-connected case.

Theorem 1.2. *Let M be an internally 4-connected binary matroid with an $M(K_5)$-minor that has no $M(K_{3,3})$-minor. Then either M has rank 4, or M is isomorphic to one of the following matroids: T_{12}, T_{12}/e, $\widetilde{T_{11}}$, $\widetilde{T_{12}}$, and $\widetilde{T_{13}}$.*

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Note that there are precisely five 3-connected rank-4 binary matroids that have a proper $M(K_5)$-minor. They are all internally 4-connected. The proof is a routine case checking. Therefore we have

Corollary 1.3. *There are precisely ten internally* 4*-connected binary matroids with a proper* $M(K_5)$*-minor that have no* $M(K_{3,3})$*-minor.*

To prove Theorem 1.2, we use Seymour's Splitter Theorem [7] and the blocking sequence techniques developed in [1] and [3]. A binary matroid will be represented by its fundamental graphs. The reader may refer to [6] or [8] for an introduction to these techniques.

The paper is constructed as follows: Section 2 contains the definitions of the matroids mentioned in Theorem 1.2 and in Section 3 we present the lemmas that are needed for the proof of Theorem 1.2.

2. Definitions

The matroid T_{12} was discovered by Kingan [4] as a splitter for the class of binary matroids with no $M(K_{3,3})$- or $M^*(K_{3,3})$-minor. The following matrix representation A_{12} of T_{12} was found by Oxley [4]:

$$A_{12} = \left[\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

It is easy to see that T_{12} is self dual. T_{12} is 4-connected. Kingan has showed that T_{12} has a transitive automorphism group. Therefore, there is a unique single-element contraction (respectively deletion) of T_{12}, denoted by T_{12}/e (respectively $T_{12}\backslash e$). T_{12}/e and $T_{12}\backslash e$ are duals of each other and both are internally 4-connected.

The matroids $\widetilde{T_{11}}$, $\widetilde{T_{12}}$, and $\widetilde{T_{13}}$ are represented by the binary matrices $\widetilde{A_{11}}$, $\widetilde{A_{12}}$, and $\widetilde{A_{13}}$, respectively:

$$\widetilde{A_{11}} = \left[\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \\ I_5 & 0 & 1 & 1 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right], \qquad \widetilde{A_{12}} = \left[\begin{array}{c|ccccccc} & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ I_5 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right],$$

$$\widetilde{A_{13}} = \left[\begin{array}{c|cccccccc} & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ I_5 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

It is routine to check that the matroids $\widetilde{T_{11}}$, $\widetilde{T_{12}}$, and $\widetilde{T_{13}}$ are internally 4-connected and have no $M(K_{3,3})$-minor. Clearly $\widetilde{T_{13}} \setminus 13 = \widetilde{T_{12}}$.

Finally let Q_{10} be the binary matroid represented by the matrix A_{10}:

$$A_{10} = \left[\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 1 & 1 \end{array}\right].$$

Note that Q_{10} is not internally 4-connected: $\{3, 4, 5, 10\}$ is a 4-element circuit and cocircuit. Q_{10} has an $M(K_{3,3})$-minor but has no $M^*(K_{3,3})$-minor.

3. Sketch of the proof

Let $\mathcal{M}$ denote the class of binary matroid with no $M(K_{3,3})$-minor. We rely heavily on the next lemma, which shows the matroid Q_{10}^* plays a similar role in $\mathcal{M}$ to what the matroid R_{12} does in the class of regular matroids.

Lemma 3.1. *Let M be a binary matroid. If M has a Q_{10}-minor and M has no $M^*(K_{3,3})$-minor, or equivalently, M has a Q_{10}^*-minor and has no $M(K_{3,3})$-minor, then M is not internally* 4-*connected.*

It is easy to see that Theorem 1.2 follows from Lemmas 3.2–3.4.

Lemma 3.2. *Let M be a matroid in $\mathcal{M}$ that has an $M(K_5)$-minor. If M is* 3-*connected and has rank at least* 5, *then M has either a T_{12}/e- or a $\widetilde{T_{11}}$-minor.*

Lemma 3.3. *Let $M \in \mathcal{M}$. If M is* 3-*connected and has a proper T_{12}/e-minor, then M is isomorphic to T_{12}, $\widetilde{T_{12}}$, or $\widetilde{T_{13}}$.*

Lemma 3.4. *Let $M \in \mathcal{M}$ be internally* 4-*connected. If M has a $\widetilde{T_{11}}$-minor and has no T_{12}/e-minor, then M is isomorphic to $\widetilde{T_{11}}$.*

Lemmas 3.2–3.4 are proved by the same techniques as used in [6,8]. The proofs are straightforward but lengthy, thus omitted. A curious reader may look at [9] for details.

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